A New Type Algorithm for the Generalized Linear Complementarity Problem Over a Polyhedral Cone In Engineering and Equilibrium Modeling

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***Abstract*— In this paper, we consider a new type algorithm for the generalized linear complementarity problem over a polyhedral cone in engineering and economic equilibrium modeling(GLCP). To this end, we first develop some equivalent reformulations of the problem under milder conditions, and then an easily computable global error bound for the GLCP is established, which can be viewed as extensions of previously known results. Based on this, we propose a new type of solution method to solve the GLCP, and show that the algorithm is global and $R-$linear convergence. Some numerical experiments of the algorithm are also reported in this paper.**

***Index Terms*— GLCP, engineering and economic equilibrium modeling, global error bound; algorithm, globally convergent, R-linear convergent**

I. INTRODUCTION

Let $F(x) = Mx + p, G(x) = Nx + q$, where $M, N \in R^{m\times n}$, $p, q \in R^m$. The generalized linear complementarity problem, abbreviated as GLCP, is to find vector $x^* \in R^n$ such that

$$F(x^*) \in \mathcal{K}, \quad G(x^*) \in \mathcal{K}^\circ, \quad F(x^*)^\top G(x^*) = 0, \quad (1)$$

where $\mathcal{K}$ is a polyhedral cone in R^m and $\mathcal{K}^\circ$ is its dual cone. Certainly, for polyhedral cone $\mathcal{K}$, there exist matrices $A \in R^{s\times m}, B \in R^{t\times m}$ such that $\mathcal{K} = \{v \in R^m \mid Av \geq 0,\ Bv = 0\}$, and its dual cone $\mathcal{K}^\circ$ assumes the following form

$$\mathcal{K}^\circ = \{u \in R^m \mid u = A^\top\lambda_1 + B^\top\lambda_2,\ \lambda_1 \in R^s_+,\ \lambda_2 \in R^t\}.$$

We denote the solution set of the GLCP by X^* and assume that it is nonempty throughout this paper.

The GLCP is a direct generalization of the classical linear complementarity problem and a special case of the generalized nonlinear complementarity problem which finds applications in engineering, economics, finance, and robust optimization operations research (Refs. [1], [2]). For example, the balance of supply and demand is central to all economic systems; mathematically, this fundamental equation in economics is often described by a complementarity relation between two sets of decision variables. Furthermore, the classical Walrasian law of competitive equilibria of exchange economies can be formulated as a generalized nonlinear complementarity problem in the price and excess demand variables ([2]), and be also found applications in contact mechanics problems(such as a dynamic rigid-body model, a discretized large displacement frictional contact problem), structural mechanics problems, obstacle problems mathematical physics, elastohydrodynamic lubrication problems, traffic equilibrium problems(such as a path-based formulation problem, a multicommodity formulation problem, network design problems), etc. Up to now, the issues of numerical methods and existence of the solution for the problem were discussed in the literature (e.g., Refs. [3]–[7]).

Among all the useful tools for theoretical and numerical treatment to variational inequalities, nonlinear complementarity problems and other related optimization problems, the global error bound, i.e., an upper bound estimation of the distance from a given point in R^n to the solution set of the problem in terms of some residual functions, is an important one due to the following reasons: First, the global error bound can not only give us a help in designing solution methods for it, e.g., providing an effective termination criteria, but also be used to analyze the convergence rate; second, it can be used in the sensitivity analysis of the problems when their data is subject to perturbation (Refs. [8], [9]). The error bound estimation for the GLCP was fully analyzed (e.g.,Refs. [10]–[12]. This motivates us to consider a new type algorithm for GLCP based on the error bound estimation for the GlCP. So, in this paper, we first develop some equivalent reformulations of the GLCP, and then we are concentrated on establishing a global error bound for the GLCP via an easily computable residual function under mild conditions which can be taken as an extension of that for GLCP, Based on this, we propose a new type of solution method to solve the GLCP, and show that the algorithm is global and $R-$linear convergence under milder assumptions. Compared with the existing solution methods in [6], [7], the conditions guaranteed for convergence are weaker in this paper.Some numeri-

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cal experiments are also reported, and indicate that this method has nice stability and high computation efficiency.

Some notations used in this paper are in order. The norm $\|\cdot\|$ denote the Euclidean 2-norm, for matrix M, the norm $\|M\|_F$ denote Frobenius-norm, i.e., $\|M\|_F = [tr(M^\top M)]^{1/2}$, where the transpose of a matrix M be denoted by $M^\top$, trace of matrix M be denoted by $tr(M^\top M)$. Without making confusion, we denote a nonnegative vector $x \in R^n$ by $x \geq 0$.

II. Several equivalent reformulation of GLCP

In this section, we will establish an equivalent reformulation of the GLCP. we first give the needed assumptions for our analysis.

Assumption 1 For the matrices A, B, M, N involved in the GLCP, we assume that the matrix UN or BM have full-column rank, where $U = (I_s, 0_{s\times t})Q^{(1,4)}, Q = (A^\top, B^\top)$, $Q^{(1,4)} \in Q\{1,4\}, Q\{1,4\}$ is set of Moore-Penrose generalized inverse matrix.

The following results is straightforward.

Theorem 1 *A point $x^* \in R^n$ is a solution of the GLCP if and only if there exist $\lambda_1^* \in R^s, \lambda_2^* \in R^t$, such that*

$$\begin{cases} AF(x^*) \geq 0, BF(x^*) = 0 \\ F(x^*)^\top G(x^*) = 0 \\ G(x^*) = A^\top \lambda_1^* + B^\top \lambda_2^* \\ \lambda_1^* \geq 0 \end{cases} \tag{2}$$

To establish an equivalent reformulation of the GLCP, we need the following conclusion in [13].

Lemma 1 *Suppose that the non-homogeneous linear equation system $Hy = b$ has solution, then $y = H^{(1,4)}b$ is minimum norm solution of its, where $H \in R^{m\times n}, b \in R^m$.*

Combining lemma 1, we can establish the following result.

Lemma 2 *Suppose that the equation $G(x) = A^\top\lambda_1 + B^\top\lambda_2$ holds, then for any $x \in R^n$, the following statements are equivalent.*

(1) There exist $\lambda_1 \in R^s_+, \lambda_2 \in R^t$ such that $G(x) = A^\top\lambda_1 + B^\top\lambda_2$,

(2)$UG(x) \geq 0, \bar{Q}G(x) = 0$, where $U = (I_s, 0_{s\times t})Q^{(1,4)}$, $\bar{Q} = QQ^{(1,4)} - I_m$, Q is defined in Assumption 1.

Proof Set

$$X_1 = \left\{ x \in R^n \;\middle|\; \begin{array}{c} G(x) = A^\top\lambda_1 + B^\top\lambda_2 \\ \text{for some } \lambda_1 \in R^s_+, \lambda_2 \in R^t \end{array} \right\},$$
$$X_2 = \{x \in R^n \mid UG(x) \geq 0, \bar{Q}G(x) = 0\}.$$

Certainly, to show the assertion, we only need to show that these two sets are equal.

For any $x \in X_1$, there exists $\lambda_1 \in R^s_+, \lambda_2 \in R^t$ such that

$$G(x) = Q\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \tag{3}$$

By Lemma 1, we obtain

$$Q^{(1,4)}G(x) = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \tag{4}$$

Combining this with (3), one has

$$(QQ^{(1,4)} - I_m)G(x) = 0. \tag{5}$$

Since $\lambda_1 \geq 0$, by (4), one has $(I_s,\ 0_{s\times t})Q^{(1,4)}G(x) \geq 0$. This, along with (5), yields that $x \in X_2$.

On the other hand, for any $x \in X_2$, let

$$\lambda_1 = (I_s, 0_{s\times t})Q^{(1,4)}G(x),$$
$$\lambda_2 = (0_{t\times s}, I_t)Q^{(1,4)}G(x).$$

Then $\lambda_1^* \in R^s_+, \lambda_2^* \in R^t$. From (5), one has

$$\begin{aligned} G(x) &= Q[Q^{(1,4)}G(x)] \\ &= (A^\top, B^\top)\begin{pmatrix} I_s & 0 \\ 0 & I_t \end{pmatrix} Q^{(1,4)}G(x) \\ &= A^\top\lambda_1 + B^\top\lambda_2, \end{aligned}$$

i.e., $x \in X_1$. The desired result follows.

From this conclusion, we can transform system (2) into a new system where neither parameter λ_1 nor parameter λ_2 is involved. Combining this with Theorem 1, the GLCP can be equivalently transformed into the following system:

$$\begin{cases} AF(x) \geq 0, \\ BF(x) = 0 \\ F(x)^\top G(x) = 0 \\ UG(x) \geq 0, \\ \bar{Q}G(x) = 0 \end{cases} \tag{6}$$

For this system, by the first and last equalities, one has

$$\begin{aligned} & F(x)^\top G(x) \\ &= F(x)^\top [QQ^{(1,4)}G(x)] \\ &= \left[\begin{pmatrix} A \\ B \end{pmatrix} F(x)\right]^\top \begin{pmatrix} I_s & 0 \\ 0 & I_t \end{pmatrix} Q^{(1,4)}G(x) \\ &= (AF(x))^\top [(I_s, 0_{s\times t})Q^{(1,4)}G(x)]. \end{aligned} \tag{7}$$

Thus, system (6) can be further rewritten as

$$\begin{cases} AF(x) \geq 0, \\ BF(x) = 0 \\ (AF(x))^\top [UG(x)] = 0 \\ UG(x) \geq 0, \\ \bar{Q}G(x) = 0 \end{cases} \tag{8}$$

Moreover, from (7), for any $x \in R^n$ such that $AF(x) \geq 0,\ UG(x) \geq 0$, it holds that $F(x)^\top G(x) \geq 0$.

Under Assumption 1, we can establish the following optimization reformulation of the GLCP based on (8) in the sense that x^* is a solution of the GLCP if and only if x^* is its global optimal solution with the objective vanishing:

$$\begin{array}{ll} \min & H(x) = (Mx+p)^\top(Nx+q) \\ & \qquad + \|B(Mx+p)\|^2 + \rho\|U(Nx+q)\|^2 \\ \text{s.t.} & x \in X \end{array} \tag{9}$$

where $\mu_1, \mu_2, \cdots, \mu_n$ are the eigenvalue of matrix $M^\top N + N^\top M$, respectively, constant

$$\rho > \frac{1}{2}\{\|\mu_1\|, \|\mu_2\|, \cdots, \|\mu_n\|\},$$
$$X = \{x \in R^n \;|A(Mx+p) \geq 0,\ U(Nx+q) \geq 0\}.$$

Without loss of generality, we assume that the matrix UN has full-column rank. Hence, the Hessian matrix of $H(x)$, i.e., $M^\top N + N^\top M + 2M^\top B^\top BM + 2\rho N^\top U^\top UN$, is positive definite, so $H(x)$ is a convex function. Furthermore, the feasible set X is a polyhedral. Thus, problem (9) is a standard strongly convex optimization. By the related optimality theory ([14]), we know that its solution set coincides with its stationary point set, i.e., with the solution set of the following variational inequality problem: find $x^* \in X$ such that

$$(x - x^*)^\top(\bar{M}x^* + \bar{q}) \geq 0, \quad \forall\, x \in X, \tag{10}$$

where

$$\bar{M} = M^\top N + N^\top M + 2M^\top B^\top BM + 2\rho N^\top U^\top UN,$$
$$\bar{q} = M^\top q + N^\top p + 2M^\top B^\top Bp + 2\rho N^\top U^\top Uq.$$

That is, the variational inequality problem (10) is also an equivalent reformulation of the GLCP.

III. The error bound for GLCP

In this section, we would establish the global error bound of the GLCP, we first need the definition of projection operator and some relate properties([15]).

For nonempty closed convex set $\Omega \subset R^n$ and any vector $x \in R^n$, the orthogonal projection of x onto Ω, i.e., $argmin\{\|y - x\| | y \in \Omega\}$, is denoted by $P_\Omega(x)$.

Lemma 3

$$(i)\quad \langle P_\Omega(u) - u, v - P_\Omega(u)\rangle \geq 0, \forall u \in R^n, v \in \Omega,$$

$$(ii)\quad \|P_\Omega(u) - P_\Omega(v)\| \leq \|u - v\|, \forall u, v \in \Omega.$$

For (10), $e(x) := x - P_\Omega[x - (\bar{M}x + \bar{q})]$ is called projection-type residual function, and let $r(x) := \|e(x)\|$. The following conclusion provides the relationship between the solution set of (10) and that of projection-type residual function([16]).

Lemma 4 *x is a solution of (10) if and only if $r(x) = 0$.*

To establish the global error bound of the GLCP, we give a conclusion which is easy to deduce.

Lemma 5 *Under Assumption 1, for any $x \in R^n$, there exist constant $\mu_{min} > 0, \mu_{max} > 0$ such that*

$$\|x\|^2\mu_{min} \leq x^\top \bar{M}x \leq \mu_{max}\|x\|^2,$$

where μ_{min}, μ_{max} are the minimum and maximum eigenvalue of matrix $\bar{M}$, respectively.

In this following, based on Lemma 3-5, we establish error bound of the GLCP which is crucial to convergence of algorithm.

Theorem 2 *Suppose that Assumption 1 holds, for any $x \in R^n$, there exist a solution x^* of (1) such that*

$$(\|I - \bar{M}\| + 1)^{-1}r(x) \leq \|x - x^*\| \leq (\|\bar{M}\| + 1)\mu_{min}^{-1}r(x).$$

Proof Since $x - e(x) = P_X[x - (\bar{M}x + \bar{q})] \in X$, by (10),

$$(x - e(x) - x^*)^\top(\bar{M}x^* + \bar{q}) \geq 0. \tag{11}$$

Combining $x^* \in X$ with Lemma 3(i), we have

$$\begin{aligned}&\langle x^* - P_X[x - (\bar{M}x + \bar{q})], P_X[x - (\bar{M}x + \bar{q})]\\ &-[x - (\bar{M}x + \bar{q})]\rangle \geq 0.\end{aligned} \tag{12}$$

Substituting $P_X[x - (\bar{M}x + \bar{q})]$ in (12) by $x - e(x)$ leads to that

$$(x - x^* - e(x))^\top[e(x) - (\bar{M}x + \bar{q})] \geq 0. \tag{13}$$

Using (11) and (13), we obtain

$$[(x - x^*) - e(x)]^\top[e(x) + \bar{M}(x^* - x)] \geq 0,$$

i.e.,

$$\begin{aligned}&(x - x^*)^\top\bar{M}(x^* - x) + e(x)^\top[(x - x^*) - \bar{M}(x^* - x)]\\ &-e(x)^\top e(x) \geq 0,\end{aligned}$$

a direct computation yields that

$$\begin{aligned}&(x - x^*)^\top\bar{M}(x - x^*)\\ &\leq e(x)^\top[(x - x^*) + \bar{M}(x - x^*)] - e(x)^\top e(x)\\ &\leq e(x)^\top[(x - x^*) + \bar{M}(x - x^*)]\\ &\leq \|e(x)\|(\|x - x^*\| + \|\bar{M}(x - x^*)\|)\\ &\leq r(x)(\|\bar{M}\| + 1)\|x - x^*\|.\end{aligned}$$

Base on Lemma 5, we have

$$(x - x^*)^\top\bar{M}(x - x^*) \geq \mu_{min}\|x - x^*\|^2,$$

so, $\mu_{min}\|x - x^*\|^2 \leq (\|\bar{M}\| + 1)r(x)\|x - x^*\|$. i.e.,

$$\|x - x^*\| \leq (\|\bar{M}\| + 1)\mu_{min}^{-1}r(x). \tag{14}$$

On the other hand, for (10), $x \in R^n, x^* \in X^*$, we have

$$\begin{aligned}r(x) &= \|e(x) - e(x^*)\|\\ &= \|x - P_X[x - (\bar{M}x + \bar{q})]\\ &\quad - x^* + P_X[x^* - (\bar{M}x^* + \bar{q})]\|\\ &\leq \|x - x^*\|\\ &\quad + \|P_X[x - (\bar{M}x + \bar{q})]\\ &\quad - P_X[x^* - (\bar{M}x^* + \bar{q})]\|\\ &\leq \|x - x^*\| + \|[x - (\bar{M}x + \bar{q})]\\ &\quad - [x^* - (\bar{M}x^* + \bar{q})]\|\\ &= \|x - x^*\| + \|(I - \bar{M})(x - x^*)\|\\ &\leq (\|(I - \bar{M})\| + 1)\|x - x^*\|.\end{aligned}$$

where the first equation is by Lemma 4, the second inequality is by Lemma 3.1(ii). Thus,

$$\|x - x^*\| \geq (\|I - \bar{M}\| + 1)^{-1}r(x), \tag{15}$$

Combining (14) with (15), then the desired result follows.

Remark. The error bound obtained in Theorem 2 is an extension of Theorem 4.1, Theorem 4.2 in [10], Theorem 3.1 in [11], Lemma 5.6 in [12] for GLCP.

IV. ALGORITHM AND CONVERGENCE

In this section, we give a new-type method to solve GLCP based on the error bound in section 3, and present the proof for its global $R-$linear convergence rate. First, we give some technical lemmas.

Lemma 6 *Under Assumption 1, let $\tau = \mu_{min}^{-1}$, we have*

$$x^\top \bar{M}x \geq \tau\|\bar{M}x\|^2. \tag{16}$$

Proof By Lemma 5, we have

$$\begin{aligned} x^\top \bar{M}x &\geq \mu_{min}\|x\|^2 \\ &\geq \mu_{min}\|\bar{M}\|^{-2}\|\bar{M}x\|^2 \\ &= \mu_{min}^{-1}\|\bar{M}x\|^2. \end{aligned}$$

Lemma 7 *Under Assumption 1, and x^* is a solution of GLCP, then*

$$\langle \bar{M}x + \bar{q}, x - x^*\rangle \geq \tau\|\bar{M}(x - x^*)\|^2, \forall x \in X,$$

where τ is defined in Lemma 6.

Proof By Lemma 6, we have

$$\begin{aligned} &\langle(\bar{M}x + \bar{q}) - (\bar{M}x^* + \bar{q}), x - x^*\rangle \\ &= (x - x^*)^\top \bar{M}(x - x^*) \\ &\geq \tau\|\bar{M}(x - x^*)\|^2. \end{aligned}$$

Since x^* is a solution of (10), so for any $x \in X$, we have $\langle \bar{M}x^* + \bar{q}, x - x^*\rangle \geq 0$, and the desired result follows.

Now, we formally state our algorithm.

Algorithm 1

Step1. Take $\varepsilon > 0$, $\gamma > \frac{\mu_{min}}{2}$, $\eta = tr((I - \bar{M})^\top(I - \bar{M}))$, $\mu_0 = \frac{tr(\bar{M})}{tr(\bar{M}^\top \bar{M})}$, and take initial point $z^0 = x^0 \in R^n$. Set $k \triangleq 0$;

Step2. Compute

$$x^{k+1} = \gamma P_X[x^k - (\bar{M}x^k + \bar{q})]; \tag{17}$$

Step3. Take $\lambda^{k+1} \in R$ such that $|(I - \lambda^{k+1}\bar{M})| \neq 0$, and

If $\eta > 1$, we take

$$\lambda_0 > \max\{\sqrt{n/[\eta tr(\bar{M}^\top \bar{M})]}, \mu_0\},$$

and

$$\max\{\sqrt{n/[\eta tr(\bar{M}^\top \bar{M})]}, \mu_0\} < \lambda^{k+1} < \eta^{-1}\lambda^k;$$

If $\eta < 1$, we take

$$\begin{aligned} &-\sqrt{n/[\eta tr(\bar{M}^\top \bar{M})]} < \lambda_0 \\ &< \min\{\sqrt{n/[\eta tr(\bar{M}^\top \bar{M})]}, \mu_0\}, \end{aligned}$$

and

$$\begin{aligned} &-\sqrt{n/[\eta tr(\bar{M}^\top \bar{M})]} < \eta^{-1}\lambda^k < \lambda^{k+1} \\ &< \min\{\sqrt{n/[\eta tr(\bar{M}^\top \bar{M})]}, \mu_0\}; \end{aligned}$$

If $\eta = 1$, we take $\lambda_0 > \mu_0$, and $\mu_0 < \lambda^{k+1} < \lambda^k$;

Let $z^{k+1} = \lambda^{k+1}(x^{k+1} - x^k) + x^k$.

Step4. If $\|x^{k+1} - x^k\| \leq \varepsilon$ stop, otherwise, go to Step 2 with $k \triangleq k + 1$.

Remark The algorithm is based on the error bound estimation of problem (1) as discussed below. Obviously, if $x^{k+1} = x^k$, combining Theorem 2, then x^k is a solution of GLCP. In the following theoretical analysis, we assume that Algorithm 1 generates an infinite sequence.

By the definition of projection operator, we can easily get that problem (17) can be equivalently reformulated as the following constrained optimization problem

$$\begin{array}{ll} \min & \gamma(x - x^k)^\top(x - x^k) \\ & +2\gamma(x - x^k)^\top(\bar{M}x^k + q) \\ \text{s.t.} & x \in X. \end{array} \tag{18}$$

Theorem 3 *Under Assumption 1, then the sequence $\{z^k\}$ converges globally to a solution of (1).*

Proof First, we prove that the convergence of sequence $\{x^k\}$.

Suppose that $x^{k+1} \neq x^k$ and let

$$\psi(\omega) = 2\gamma(\omega - x^*)^\top(\bar{M}x^* + q) + \gamma\|\omega - x^*\|^2,$$

where $x^* \in X^*$. Since $\bar{M}$ is positive definite, combining the definition of $\psi(\omega)$ with (10), we know that

$$\psi(\omega) \geq \gamma\|\omega - x^*\|^2 \geq 0. \tag{19}$$

We claim that the nonnegative sequence $\{\psi(x^k)\}$ is monotonically decreasing. In fact, since problem (18) can be equivalently reformulated as the following variational inequalities, for any $\omega \in X$, we have

$$\langle x^{k+1} - x^k, \omega - x^{k+1}\rangle + \langle \bar{M}x^k + \bar{q}, \omega - x^{k+1}\rangle \geq 0, \tag{20}$$

it follows that

$$\begin{aligned} &\psi(x^k) - \psi(x^{k+1}) \\ &= \gamma(x^k - x^*)^\top(x^k - x^*) + 2\gamma\langle \bar{M}x^* + \bar{q}, x^k - x^*\rangle \\ &\quad - \gamma(x^{k+1} - x^*)^\top(x^{k+1} - x^*) \\ &\quad - 2\gamma\langle \bar{M}x^* + \bar{q}, x^{k+1} - x^*\rangle \\ &= \gamma(x^k)^\top x^k - \gamma(x^*)^\top x^* - 2\gamma\langle x^*, x^k - x^*\rangle \\ &\quad - \gamma(x^{k+1})^\top x^{k+1} + \gamma(x^*)^\top x^* \\ &\quad + 2\gamma\langle x^*, x^{k+1} - x^*\rangle + 2\gamma\langle \bar{M}x^* + \bar{q}, x^k - x^{k+1}\rangle \\ &= \gamma(x^k)^\top x^k - \gamma(x^{k+1})^\top x^{k+1} + 2\gamma\langle x^*, x^{k+1} - x^k\rangle \\ &\quad + 2\gamma\langle \bar{M}x^* + \bar{q}, x^k - x^{k+1}\rangle \\ &= \gamma(x^k)^\top x^k - \gamma(x^{k+1})^\top x^{k+1} \\ &\quad - 2\gamma\langle x^{k+1}, x^k - x^{k+1}\rangle \\ &\quad + 2\gamma\langle x^{k+1} - x^*, x^k - x^{k+1}\rangle \\ &\quad + 2\gamma\langle \bar{M}x^* + \bar{q}, x^k - x^{k+1}\rangle \\ &= \gamma(x^k - x^{k+1})^\top(x^k - x^{k+1}) \\ &\quad + 2\gamma\langle x^{k+1} - x^k, x^* - x^{k+1}\rangle \\ &\quad + 2\gamma\langle \bar{M}x^* + \bar{q}, x^k - x^{k+1}\rangle \\ &\geq \gamma(x^k - x^{k+1})^\top(x^k - x^{k+1}) \\ &\quad - 2\gamma\langle \bar{M}x^k + \bar{q}, x^* - x^{k+1}\rangle \\ &\quad + 2\gamma\langle \bar{M}x^* + \bar{q}, x^k - x^{k+1}\rangle \\ &= \gamma(x^k - x^{k+1})^\top(x^k - x^{k+1}) \\ &\quad + 2\gamma\langle \bar{M}x^k + \bar{q}, x^k - x^*\rangle \\ &\quad - 2\gamma\langle \bar{M}x^k + \bar{q}, x^k - x^{k+1}\rangle \\ &\quad + 2\gamma\langle \bar{M}x^* + \bar{q}, x^k - x^{k+1}\rangle \end{aligned}$$

$$
\begin{aligned}
&\geq \gamma(x^k - x^{k+1})^\top (x^k - x^{k+1}) + 2\gamma\tau\|\bar{M}(x^k - x^*)\|^2 \\
&\quad - \langle 2\gamma\bar{M}(x^k - x^*), x^k - x^{k+1}\rangle \\
&\geq \gamma\|x^k - x^{k+1}\|^2 + 2\gamma\tau\|\bar{M}(x^k - x^*)\|^2 \\
&\quad - (2\gamma\tau\|\bar{M}(x^k - x^*)\|^2 + \tfrac{1}{2\tau}\|x^k - x^{k+1}\|^2) \\
&\geq \gamma\|x^k - x^{k+1}\|^2 - \tfrac{1}{2\tau}\|x^k - x^{k+1}\|^2,
\end{aligned}
$$

where the first inequality is obtained by letting $\omega = x^*$ in inequality (20); the second inequality follows from Lemma 7 by letting $x = x^k$, and the last inequality follows from the Cauchy-Schwarz inequality.

Since $\gamma > \frac{1}{2\tau}$, we conclude that $\psi(x_k) - \psi(x_{x+1}) > 0$ and thus the nonnegative sequence $\{\psi(x^k)\}$ is strictly decreasing and convergent. Consequently,

$$\lim_{k\to\infty} \|x^k - x^{k+1}\| = 0. \tag{21}$$

Moreover, $\{\psi(x^k)\}$ is bounded since it is convergent, and so is $\{x^k\}$ according to (19). Combining Theorem 2 with (21), we have

$$dist(x^k, X^*) \leq (\|\bar{M}\|+1)\mu_{min}^{-1}\|x^{k+1}-x^k\| \to 0\ (k\to\infty),$$

where $dist(x^k, X^*) = \|x^k - \bar{x}^k\|, \bar{x}^k$ denote the closest solution to x^k.

Since $\{x^k\}$ is bounded, let $\{x^{k_i}\}$ be a subsequence of $\{x^k\}$ and converges to $\bar{x}$, for $\{x^{k_i}\}$, we also have

$$dist(x^{k_i}, X^*) \leq (\|\bar{M}\| + 1)\mu_{min}^{-1}\|x^{k_i+1} - x^{k_i}\| \to 0 \quad (i\to\infty),$$

by Theorem 2, we get $\bar{x}$ is a solution of GLCP. Since $\{\psi(x^k)\}$ converges, substituting x^* in (19) by $\bar{x}$ leads to that $\psi(x^{k_i}) \to 0(i \to \infty)$. Thus, $\psi(x^k) \to 0(k \to \infty)$. Using (19) again, we know that the sequence $\{x^k\}$ converges to $\bar{x}$.

Secondly, we prove that the sequence $\{z^k\}$ converges globally to $\bar{x}$. For any $\varepsilon > 0$, we have

$$
\begin{aligned}
\|z^{k+1} - \bar{x}\| &= \|\lambda^{k+1}x^{k+1} + (1-\lambda^{k+1})x^k - \bar{x}\| \\
&\leq \|\lambda^{k+1}x^{k+1} - \lambda^{k+1}\bar{x}\| \\
&\quad +\|(1-\lambda^{k+1})x^k - (1-\lambda^{k+1})\bar{x}\| \\
&\leq \lambda^{k+1}\varepsilon + (1-\lambda^{k+1})\varepsilon \\
&= \varepsilon.
\end{aligned}
$$

Lemma 8 *Under Assumption 1, according to the acceptance rule of λ^{k+1} in Algorithm 1, let*

$$\beta := \|(I - \lambda^k\bar{M})^{-1}\|_F\|(I - \lambda^{k+1}\bar{M})\|_F\|(I - \bar{M})\|_F,$$

then $\beta < 1$.

Proof Since

$$
\begin{aligned}
&\|(I - \lambda^{k+1}\bar{M})\|_F^2 \\
&= tr((I - \lambda^{k+1}\bar{M})^\top(I - \lambda^{k+1}\bar{M})) \\
&= tr(I - \lambda^{k+1}(\bar{M} + \bar{M}^\top) + (\lambda^{k+1})^2(\bar{M}^\top\bar{M})) \\
&= n - 2\lambda^{k+1}tr(\bar{M}) + (\lambda^{k+1})^2tr(\bar{M}^\top\bar{M}),
\end{aligned}
$$

Similarly,

$$\|(I - \lambda^k\bar{M})\|_F^2 = n - 2\lambda^k tr(\bar{M}) + (\lambda^k)^2 tr(\bar{M}^\top\bar{M})\ ,$$

then we have

$$
\begin{aligned}
\beta &= \|(I - \lambda^k\bar{M})^{-1}\|_F\|(I - \lambda^{k+1}\bar{M})\|_F\|(I - \bar{M})\|_F \\
&= \tfrac{(tr((I-\bar{M})^\top(I-\bar{M})))(n-2\lambda^{k+1}tr(\bar{M})+(\lambda^{k+1})^2tr(\bar{M}^\top\bar{M}))}{n-2\lambda^k tr(\bar{M})+(\lambda^k)^2tr(\bar{M}^\top\bar{M})} \\
&= \eta\tfrac{f(\lambda^{k+1})}{f(\lambda^k)},
\end{aligned}
\tag{22}
$$

where $\eta = tr((I - \bar{M})^\top(I - \bar{M}))$,

$$f(x) = n - 2xtr(\bar{M}) + x^2tr(\bar{M}^\top\bar{M}).$$

According to the acceptance rule of λ^{k+1} in the third step of Algorithm 1, we have that the following conclusion hold.

(1) If $\eta > 1$, we take

$$\max\{\sqrt{n/[\eta tr(\bar{M}^\top\bar{M})]}, \mu_0\} < \lambda^{k+1} < \eta^{-1}\lambda^k,$$

where $\mu_0 = (tr(\bar{M}))/(tr(\bar{M}^\top\bar{M}))$. A direct computation yields that

$$\eta f(\lambda^{k+1}) < f(\eta\lambda^{k+1}) < f(\eta\eta^{-1}\lambda^k) = f(\lambda^k), \tag{23}$$

where the first inequality is obtained by

$$\lambda^{k+1} > \sqrt{n/[\eta tr(\bar{M}^\top\bar{M})]},$$

the second inequality follows from the fact that $f(x)$ is monotonic increasing when $\lambda^{k+1} > \mu_0$. Combining (23) with (22), we have $\beta < 1$.

(2) If $0 < \eta < 1$, we take

$$
\begin{aligned}
&-\sqrt{n/[\eta tr(\bar{M}^\top\bar{M})]} < \eta^{-1}\lambda^k < \lambda^{k+1} \\
&< \min\{\sqrt{n/[\eta tr(\bar{M}^\top\bar{M})]}, \mu_0\}.
\end{aligned}
$$

A direct computation yields that

$$\eta f(\lambda^{k+1}) < f(\eta\lambda^{k+1}) < f(\eta\eta^{-1}\lambda^k) = f(\lambda^k), \tag{24}$$

where the first inequality is obtained by

$$-\sqrt{n/[\eta tr(\bar{M}^\top\bar{M})]} < \lambda^{k+1} < \sqrt{n/[\eta tr(\bar{M}^\top\bar{M})]},$$

the second inequality follows from the fact that $f(x)$ is monotonic decreasing when $\lambda^{k+1} < \mu_0$. Combining (24) with (22), we have $\beta < 1$.

(3) If $\eta = 1$, we take $\mu_0 < \lambda^{k+1} < \lambda^k$, then $f(x)$ is monotonic increasing. A direct computation yields that $f(\lambda^{k+1}) < f(\lambda^k)$. Combining this with (22), we have $\beta < 1$.

Theorem 4 *Under Assumption 1, the sequence $\{z^k\}$ globally converges to a solution of GLCP $R-$linearly.*

Proof For (18), by using KKT condition, there exist $u_1^k \in R_+^s, u_2^k \in R_+^s$ such that

$$2\gamma(x^{k+1} - x^k) + 2\gamma(\bar{M}x^k + \bar{q}) = (AM)^\top u_1^k + (UN)u_2^k,$$

i.e.,

$$x^{k+1} = (I - \bar{M})x^k + c^k \tag{25}$$

where $c^k = \frac{1}{2\gamma}((AM)^\top u_1^k + (UN)u_2^k) - \bar{q}$. Using (25), we have

$$
\begin{aligned}
z^{k+1} &= \lambda^{k+1}x^{k+1} + (1-\lambda^{k+1})x^k \\
&= (I - \lambda^{k+1}\bar{M})x^k + \lambda^{k+1}c^k.
\end{aligned}
\tag{26}
$$

Using the technique of (26), we can also get

$$z^k = (I - \lambda^k\bar{M})x^{k-1} + \lambda^k c^{k-1}. \tag{27}$$

By the definition of z^k in Algorithm 1, we have

$$z^k = \lambda^k x^k + (1-\lambda^k)x^{k-1}. \tag{28}$$

Pre-multiplying (28) by $(I-\lambda^k\bar{M})$, one has

$$\begin{array}{l}(I-\lambda^k\bar{M})z^k\\=\lambda^k(I-\lambda^k\bar{M})x^k+(1-\lambda^k)(I-\lambda^k\bar{M})x^{k-1}.\end{array}\quad(29)$$

Pre-multiplying (27) by $(1-\lambda^k)$, one has

$$\begin{array}{l}(1-\lambda^k)z^k\\=(1-\lambda^k)(I-\lambda^k\bar{M})x^{k-1}+(1-\lambda^k)\lambda^k c^{k-1}.\end{array}\quad(30)$$

By (29)-(30), we have

$$(I-\bar{M})z^k=(I-\lambda^k\bar{M})x^k-(1-\lambda^k)c^{k-1}.\quad(31)$$

Pre-multiplying (31) by $(I-\lambda^{k+1}M)$, one has

$$\begin{array}{l}(I-\lambda^{k+1}\bar{M})(I-\bar{M})z^k\\=(I-\lambda^{k+1}\bar{M})(I-\lambda^k\bar{M})x^k\\-(1-\lambda^k)(I-\lambda^{k+1}\bar{M})c^{k-1}.\end{array}\quad(32)$$

Pre-multiplying (26) by $(I-\lambda^kM)$, one has

$$\begin{array}{l}(I-\lambda^k\bar{M})z^{k+1}\\=(I-\lambda^k\bar{M})(I-\lambda^{k+1}\bar{M})x^k\\+\lambda^{k+1}(I-\lambda^k\bar{M})c^k.\end{array}\quad(33)$$

A direct computation yields that

$$(I-\lambda^{k+1}\bar{M})(I-\lambda^k\bar{M})=(I-\lambda^k\bar{M})(I-\lambda^{k+1}\bar{M}).$$

Using (33)-(32), and combining the third step of Algorithm 1, we can obtain

$$z^{k+1}=(I-\lambda^k\bar{M})^{-1}[(I-\lambda^{k+1}\bar{M})(I-\bar{M})z^k+\Delta_k],\quad(34)$$

where $\Delta_k=\lambda^{k+1}(I-\lambda^k\bar{M})c^k+(1-\lambda^k)(I-\lambda^{k+1}M)c^{k-1}$. Since the sequence $\{x^k\}$ is bounded according to the proof above, combining (25), there exist constant $\eta>0,\ \alpha>0$ such that$\|c^k\|\le\eta$, and

$$\|(I-\lambda^k\bar{M})^{-1}[(I-\lambda^{k+1}\bar{M})(I-\bar{M})\bar{x}+\Delta^k]-\bar{x}\|\le\alpha.\quad(35)$$

By using (34) and (35), we obtain

$$\begin{array}{l}\|z^{k+1}-\bar{x}\|\\=\|(I-\lambda^k\bar{M})^{-1}(I-\lambda^{k+1}\bar{M})(I-\bar{M})(z^k-\bar{x})\|\\+\|(I-\lambda^k\bar{M})^{-1}[(I-\lambda^{k+1}\bar{M})(I-\bar{M})\bar{x}+\Delta^k]-\bar{x}\|\\\le\|(I-\lambda^k\bar{M})^{-1}(I-\lambda^{k+1}\bar{M})(I-\bar{M})\|_F\|(z^k-\bar{x})\|+\alpha\\\le\beta\|(z^k-\bar{x})\|+\alpha\\\le\beta^2\|(z^{k-1}-\bar{x})\|+\beta\alpha\\\cdots\cdots\\\le\beta^{k+1}\|(z^0-\bar{x})\|+\beta^k\alpha\\=\beta^{k+1}[\|(z^0-\bar{x})+\frac{\alpha}{\beta}\|].\end{array}$$

where $\beta=\|(I-\lambda^k\bar{M})^{-1}\|_F\|(I-\lambda^{k+1}\bar{M})\|_F\|(I-\bar{M})\|_F$, combining Lemma 8, we have $\beta<1$, thus, the sequence $\{z^k\}$ converges to a solution of (1) at globally and $R-$linear convergence rate.

V. Computational Experiments

In the following, we will implement Algorithm 1 in Matlab and run it on a Pentium IV computer. Throughout our computation, **Iter** denotes the number of iterations.

$$d(n)=\|z^k-z^{k-1}\|,$$

where k denotes the number of iterations when the algorithm terminates, and γ denotes the parameter we take.

Example 1 *This problem is a linear complementarity problem (LCP) used by Harker and Pang ([17]), in which* $F(x)=Mx+p$, *where*

$$M=\begin{pmatrix}1&2&2&\cdots&2&2\\0&1&2&\cdots&2&2\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&1&2\\0&0&0&\cdots&0&1\end{pmatrix},\quad p=\begin{pmatrix}-1\\-1\\\vdots\\-1\end{pmatrix}.$$

For this problem, we take the initial point $x^0=(1,1,\cdots,1)^\top$ and parameter $\varepsilon=10^{-18}$, Harker and Pang ([17]) used the damped-Newton method (DNA), and He and Yang ([18]) used the projection and contraction method(PCA), and Han ([19]) used a hybrid generalized proximal method(HGPA), and Wang ([7]) used the Newton-type method (NTA). The results for the above four methods and several values of the dimensions n are summarized in Table 1. In Table 2, we summarize the results of our algorithm for several values of dimensions n. From Table 1 and Table 2, we can conclude that our algorithm excels the other four methods, and it has nice stability and high computation efficiency.

To illustrate the stability of our algorithm, under $n=64$ and the initial point x^0 is produced randomly in (0,5), we use it to solve example 1, and the results are listed in Table 3. Table 2 and Table 3 indicate that our algorithm is not sensitive to the change of initial point and the dimension, thus it is very stable.

TABLE 1.

Numerical Results by DNA, PCA, HGPA, NTA for Example 1

Dimension	8	16	32	64	128
DAN iter.num.	9	20	72	208	>300
PCA iter.num.	24	25	27	29	32
HGPA iter.num.	7	9	10	11	13
NAT iter.num.	13	12	18	99	99

TABLE 2.

Numerical Results of Our Algorithm for Example 1

Dimension	8	16	32	64	128	256
iter.num.	5	6	7	8	9	10
γ	1.1	1.2	1.2	1.2	1.2	1.2
$d(n)$	0	0	0	0	0	0

TABLE 3.

Numerical Results of Our Algorithm by Random Initial Point for Example 1

Trial	1	2	3	4	5
iter.num.	9	9	9	9	9
$d(n)$	0	0	5.4738×10^{-48}	0	0
Trial	6	7	8	9	10
iter.num.	9	9	9	9	9
$d(n)$	0	0	0	0	0

Example 2 *This example is a general variational inequality used by Noor and Wang ([20]). Let $F(x) = Mx + p$, where*

$$M = \begin{pmatrix} 4 & -2 & 0 & \cdots & 0 \\ 1 & 4 & -2 & \cdots & 0 \\ 0 & 1 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \\ 0 & 0 & 0 & \cdots & 4 \end{pmatrix}, \quad p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix},$$

the domain set $C = \{x \in R^n \mid 0 \le x_i \le 1,\, i = 1, 2, \cdots, n\}$.

By Theorem 2.2 in [12], the problem in Example 2 can be rewritten as the following GLCP:

$$x \in C, F(x) \in C^\circ, x^\top F(x) = 0,$$

where C° is dual cone of set C.

For this test problem, table 4 give the results for this example with starting point $x^0 = -M^{-1}p$ and parameter $\varepsilon = 10^{-15}, \gamma = 6$ for different dimensions n. Compared with the results of Table 4.2 in [20], we can conclude that our algorithm excels method in [20].

TABLE 4.

The Numerical Experiments Result for Example 2

Dimension	10	20	50
iter.num.	1	1	1
CPU Time	0	0.0310	0.1880
$d(n)$	0	0	0
Dimension	80	100	200
iter.num.	1	1	1
CPU Time	0.6570	1.2500	14.6410
$d(n)$	0	0	0

Example 3 *This example is GLCP used by Wang ([21]). Let*

$$x \in \mathcal{K},\ G(x) = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} x + \begin{pmatrix} -8 \\ -6 \\ -4 \end{pmatrix} \in \mathcal{K}^0,$$

where $\mathcal{K} = \{x \in R^3 \mid Ax \ge p\}$, where

$$A^\top = -\begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix},$$

$$p^\top = -\begin{pmatrix} 3, & 1, & 1, & 1, & 0, & 0, & 0 \end{pmatrix}$$

It is easy to verify that its polar cone $\mathcal{K}^\circ$

Table 5 list the numeric results of Algorithm 1 with different starting points. We take parameter $\varepsilon = 10^{-15}, \gamma = 6.5$. Compared with the results of Table 2 in [21], we can conclude that our algorithm excels the method in [21], and it has nice stability

To illustrate the stability of our algorithm, the initial point x^0 is produced randomly in (0,5), we use it to solve example 3, and the results are listed in Table 6. Table 5 and Table 6 indicate that our algorithm is not sensitive to the change of initial point, thus it is very stable.

TABLE 5.

The Numerical Experiments Result for Example 3

Starting point	**Iter.num.**	$\mathbf{d(n)}$	**CPU time**
$(0,0,0)^\top$	1	4.5776×10^{-16}	0.0160
$(1,0,0)^\top$	2	0	0
$(0,1,0)^\top$	1	7.1089×10^{-16}	0.0160
$(0,0,1)^\top$	2	0	0

TABLE 6.

The Numerical Experiments Result for Random Intinal Point of Example 3 (1e-16)

Trial	1	2	3	4	5
Iter.num.	2	1	1	1	1
$\mathbf{d(n)}$	0	6.3777	8.1584	6.3777	8.1584
CPU time	0	0.0160	0	0	0
Trial	6	7	8	9	10
Iter.num.	1	2	1	1	1
$\mathbf{d(n)}$	8.1584	0	8.1584	4.1541	6.8439
CPU time	0.0150	0	0.0150	0	0

VI. Conclusion

We present an easily computable global error bound for the GLCP, which can be viewed as extensions of previously known results. Based on this, we also propose a new type of solution method to solve the GLCP, and show that the algorithm is global and $R-$linear convergence.

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